Exercises

Exercise 1. Determine whether the following are linearly independent or linearly dependent.

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}$$

We put these into a matrix and row–reduce:

$$\begin{pmatrix}1 & -1\\1 & 0\end{pmatrix} \xrightarrow{R2:=R2-R1} \begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}.$$

There are no free variables, so this means there is only one solution to:

$$\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

so they are linearly independent.
Also, the determinant is 1 which shows they are linearly independent as well.

Exercise 2. Determine whether the following are linearly independent or linearly dependent.

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}$$

There are several related ways to think about this. First, if we put them in a matrix we get:

$$\begin{pmatrix}1 & -1 & 1\\1 & 0 & -1\end{pmatrix}$$

we know (with out row reducing) that there will be free variables when put into echelon form. This is because there are only two rows so there can be a maximum of 2 pivot columns. Since there are three columns, at least one must not be a pivot column. So, the vectors are linearly dependent.
We can also actually do the reduction:

$$\begin{pmatrix}1 & -1 & 1\\1 & 0 & -1\end{pmatrix} \xrightarrow{R2:=R2-R1} \begin{pmatrix}1 & -1 & 1\\0 & 1 & -2\end{pmatrix}.$$

This is in echelon form and the third column isn't a pivot column, so they are linearly dependent.

Exercise 3. Determine whether the following are linearly independent or linearly dependent.

$$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$$

Putting them in to a matrix we compute the determinant:

$$\det\begin{pmatrix}1 & 1 & 0\\0 & 0 & 1\\1 & -1 & 0\end{pmatrix} = 1\det\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} - 1\det\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = 1-(-1) = 2.$$

Since the determinant is not 0, the vectors are linearly independent.
Another way would be to row reduce:

$$\begin{pmatrix}1 & 1 & 0\\0 & 0 & 1\\1 & -1 & 0\end{pmatrix} \xrightarrow{R2\leftrightarrow R3} \begin{pmatrix}1 & 1 & 0\\1 & -1 & 0\\0 & 0 & 1\end{pmatrix} \xrightarrow{R2:=(-1/2)(R2-R3)} \begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}.$$

This is in echelon form and there are no free variables; so there is only one solution to

$$\begin{pmatrix}1 & 1 & 0\\1 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

So the vectors are linearly independent.

Exercise 4. Determine whether the following are linearly independent or linearly dependent.

$$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}$$

Putting them into a matrix and finding the determinant:

$$\det\begin{pmatrix}1 & 1 & 2\\0 & 0 & 0\\1 & -1 & 0\end{pmatrix} = 1\det\begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix} - 1\det\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + 2\det\begin{pmatrix}0 & 0\\1 & -1\end{pmatrix} = 0.$$

So they are linearly dependent.

Exercise 5. Determine whether the following are linearly independent or linearly dependent.

$$\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\\2\end{pmatrix} \begin{pmatrix}1\\1\\-2\\-2\end{pmatrix}, \begin{pmatrix}4\\1\\0\\2\end{pmatrix}$$

We put them into a matrix and row reduce (swapping the second and fourth rows):

$$\begin{pmatrix}1&2&1&4\\1&2&-2&2\\1&1&-2&0\\0&0&1&1\end{pmatrix} \xrightarrow[R2:=R2-R1]{R3:=-(R3-R1)} \begin{pmatrix}1&2&1&4\\0&0&-3&-2\\0&1&3&4\\0&0&1&1\end{pmatrix} \xrightarrow{R2\leftrightarrow R3} \begin{pmatrix}1&2&1&4\\0&1&3&4\\0&0&-3&-2\\0&0&1&1\end{pmatrix} \xrightarrow{R3\leftrightarrow R4} \begin{pmatrix}1&2&1&4\\0&1&3&4\\0&0&1&1\\0&0&-3&-2\end{pmatrix}$$

$$\xrightarrow{R4=R4+3R3} \begin{pmatrix}1&2&1&4\\0&1&3&4\\0&0&1&1\\0&0&0&1\end{pmatrix}.$$

This matrix is in echelon form and there are no free variables. So there is only one solution to:

$$\begin{pmatrix}1&2&1&4\\1&2&-2&2\\1&1&-2&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

and so the vectors are linearly independent.

Exercise 6. Determine whether the following are linearly independent or linearly dependent.

$$\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\\2\end{pmatrix} \begin{pmatrix}3\\1\\3\\3\end{pmatrix}, \begin{pmatrix}4\\0\\3\\4\end{pmatrix}$$

We row reduce (note: below we move the third row to the bottom):

$$\begin{pmatrix}1&2&3&4\\1&1&3&3\\1&2&3&4\\0&0&1&0\end{pmatrix} \xrightarrow[R2:=R2-R1]{R3:=R3-R3} \begin{pmatrix}1&2&3&4\\0&-1&0&-1\\0&0&0&0\\0&0&1&0\end{pmatrix} \xrightarrow{R3\leftrightarrow R4} \begin{pmatrix}1&2&3&4\\0&-1&0&-1\\0&0&1&0\\0&0&0&0\end{pmatrix}.$$

This is in echelon form; the fourth variable is a free variable so the vectors are linearly dependent.

Exercise 7. For which values of t are the following vectors linearly independent?

$$\{e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}, e^t \begin{pmatrix} \sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix}\}.$$

We are going to put these into a matrix and find the determinant. Note that the determinant will be a function of t and so we will find the values of t that make it zero; those will be the values for which the vectors are linearly dependent and will be linearly independent for all other values of t.

$$\begin{aligned} \det \begin{pmatrix} e^t \cos 2t & e^t \sin 2t \\ e^t(\cos 2t + \sin 2t) & e^t(-\cos 2t + \sin 2t) \end{pmatrix} &= e^{2t}(-\cos^2 2t + \cos 2t \sin 2t) - e^{2t}(\sin 2t \cos 2t - \sin^2 2t) \\ &= e^{2t}(-\cos^2 2t - \sin^2 2t + \cos 2t \sin 2t - \sin 2t \cos 2t) \\ &= -e^{2t}. \end{aligned}$$

Note that this is never 0. So the vectors are always linearly independent.

Exercise 8. For which values of t are the following vectors linearly independent?

$$\{e^{-t}\begin{pmatrix}2\cos 2t\\ \sin 2t\end{pmatrix}, e^{-t}\begin{pmatrix}-2\sin 2t\\ \cos 2t\end{pmatrix}\}.$$

We are going to put these into a matrix and find the determinant. Note that the determinant will be a function of t and so we will find the values of t that make it zero; those will be the values for which the vectors are linearly dependent and will be linearly independent for all other values of t.

$$\begin{aligned}\det\begin{pmatrix}2e^{-t}\cos 2t & -2e^{-t}\sin 2t\\ e^{-t}\sin 2t & e^{-t}\cos 2t\end{pmatrix} &= e^{-2t}(2\cos^2 2t + 2\sin^2 2t) - e^{2t}(\sin 2t\cos 2t - \sin^2 2t)\\ &= 2e^{-2t}.\end{aligned}$$

Note that this is never 0. So the vectors are always linearly independent.