Linear Algebra Workshop

① Systems of Equations

② Matrices

③ Vector spaces

④ Linear Transformations

⑤ Inner product spaces

Lecture 1

$\mathbb{R} = \{\text{real numbers}\}$
$\mathbb{C} = \{\text{complex numbers}\}$ } scalars (field)

Tuples of real numbers

"belongs to"

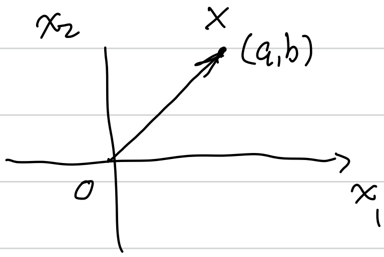
$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$$

"such that"

Set-builder

"Regard each $(x_1, \dots, x_n)$ (n-tuple) as a vector"

Ex The coordinate vector of a point on the plane is an element of $\mathbb{R}^2$.



$\vec{OX}$: the vector represented by (a,b).

Def'n We can define two operations on $\mathbb{R}^n$:

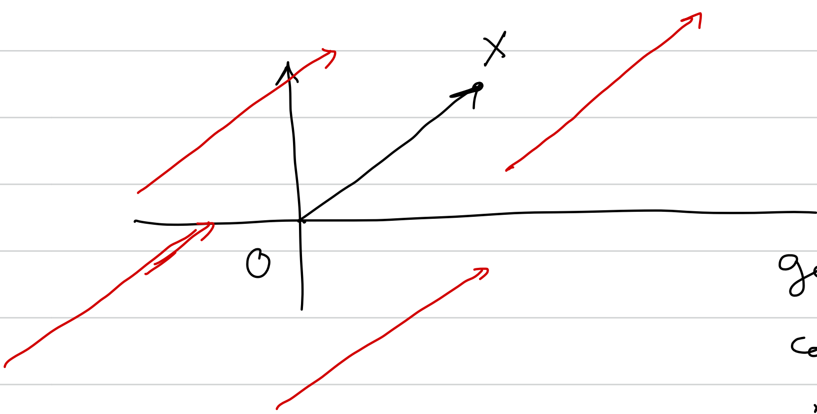
is defined as

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) := (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$$

$$\alpha (x_1, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

scalar multiplications $\alpha \in \mathbb{R}$

Ex $(1,2,3) + 2(-1,2,1) = (1,2,3) + (-2,4,2) = (-1,6,5)$



geometric vector: collection of all arrows parallel to a given $\vec{OX}$ with same direction and magnitude.

Linear equation

$$x + y = 1$$

1 linear equation in 2 variables.

Goal: Solve it!

Sol'n set $y = s$, where s is any real number (parameter, y is free variable)

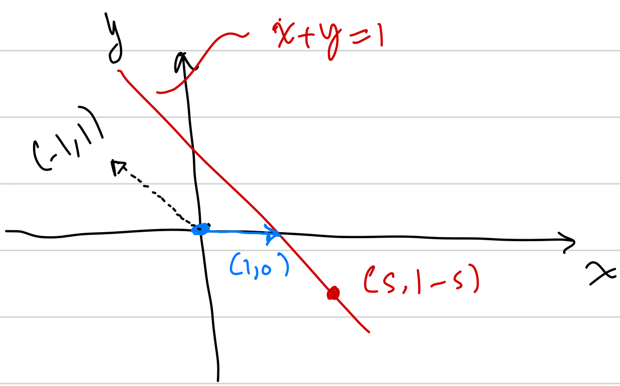
$$x = 1 - s$$

$(1-s, s)$ is a solution

$\uparrow$ x, $\uparrow$ y

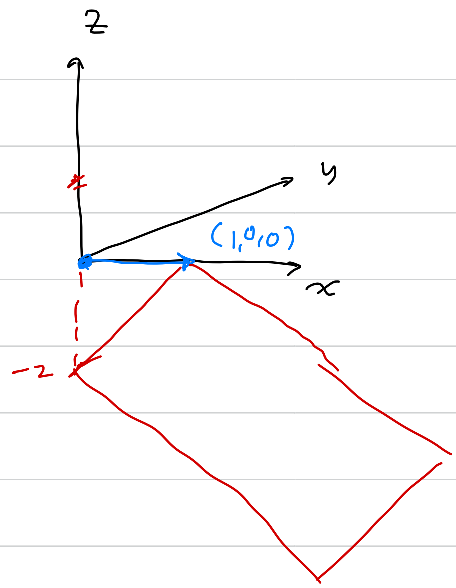
Solution Set = {all solutions}

$$S = \{ (1-s, s) \mid s \in \mathbb{R} \} \subset \mathbb{R}^2$$



$$(1-s, s) = (1,0) + (-s, s)$$

$$= \underbrace{(1,0)}_{\text{move us from 0 to the point (1,0) on the solution set}} + \underbrace{s(-1,1)}_{\text{moving inside the solution set}}$$



<u>Ex</u> $2x + y - 4z = 2$

let

$z = s$, $y = t$ (free)

$$2x = 2 - y + 4z$$

$$x = 1 - \frac{y}{2} + 2z = 1 - \frac{t}{2} + 2s$$

$$S = \left\{ \left(1 - \frac{t}{2} + 2s,\ t,\ s\right) \;\middle|\; s, t \in \mathbb{R} \right\} \quad (\subset \mathbb{R}^3)$$

↑

solution set

$$\left(1 - \frac{t}{2} + 2s,\ t,\ s\right) = (1, 0, 0) + \left(-\frac{t}{2},\ t,\ 0\right) + (2s,\ 0,\ s)$$

$$= (1,0,0) + t\left(-\frac{1}{2}, 1, 0\right) + s\,(2, 0, 1)$$

takes us to the solution set

Parallel to the plane of solutions

then this vector moves along the solution set.

Two linear equations

$$\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$$

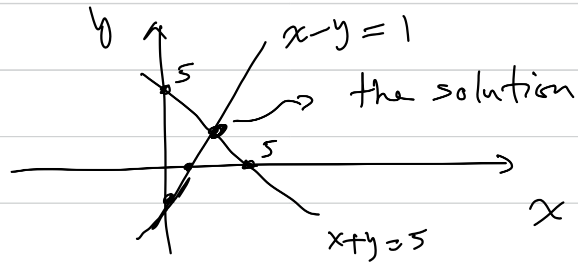
Subtracting the first from the second equations (eliminates x)

$$\begin{cases} x + y = 5 \\ \quad -2y = -4 \end{cases} \Rightarrow \underline{y = 2}$$

Substituting backward : $x + 2 = 5 \Rightarrow \underline{x = 3}$

$S = \{(3,2)\}$ Singleton : "with an element"

$\boxed{\#\text{ solutions} = 1}$



the solution $(3,2)$

__Ex__

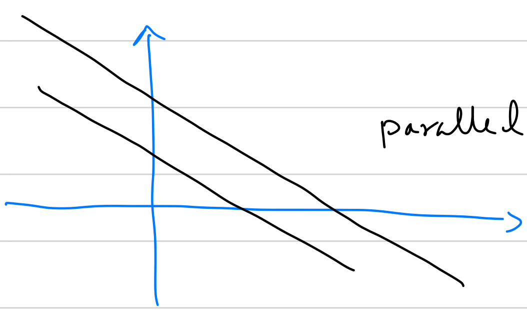
$$\begin{cases} x + 3y = 1 & (1) \\ 2x + 6y = 5 & (2) \end{cases}$$

multiply (1) by $\underline{\underline{-2}}$ add to (1)

$$\begin{cases} x + 3y = 1 \\ \quad 0 = 3 \end{cases}$$ impossible

$S = \phi$ empty set

the system is inconsistent



parallel lines, no points in common

$\underline{\#\text{ solutions}} = 0$

$\underline{\text{Ex}}$ $\begin{cases} x + 2y = 3 \\ 4x + 8y = 12 \end{cases} \rightsquigarrow \begin{cases} x + 2y = 3 \\ 0 = 0 \end{cases}$

\# solutions = infinity.

\# solutions : $0, 1, \infty$

System of equations

m linear equations in n variables $x_1, \ldots, x_n$.

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\
&\vdots \\
a_{m1} x_1 + \cdots\cdots + a_{mn} x_n &= b_m
\end{aligned}
$$

a_{ij} : coefficients $\quad 1 \le i \le m, \quad 1 \le j \le n$

b_i : constant terms

Def'n

A solution for a system as above is an n-tuple $(x_1, \ldots, x_n) \in \mathbb{R}^n$ that satisfies all equations of the system.

If there is at least one solution, we call the system consistent. otherwise, inconsistent.

Two systems of equations are equivalent if if they have the same solution set.

① $\begin{cases} x+y=5 \\ x-y=1 \end{cases}$ is equivalent to $\begin{cases} x+y=5 \\ -2y=-4 \end{cases}$

$S_1=\{(3,2)\}$

$\begin{cases} x+y=5 \\ 0=0 \end{cases}$ not equivalent to the above

$\hookrightarrow$ $S_2=\{(5-s,s) \mid s\in\mathbb{R}\}$

$S_1 \subsetneq S_2$

Elementary operations

① Switching two equations

② multiplying an equation by a nonzero scalar.

③ adding a scalar multiple of one equation to another.

Proposition Applying an elementary operation produces an equivalent system.

Definition A system is **homogeneous** if the constant terms are all $\underline{0}$.

$$\begin{cases} x+y-z=0 \\ 2x-2y+z=0 \end{cases}$$

always consistent.

b/c $\underbrace{(0,0,\dots,0)}_{n \text{ times}} \in S$

$(0,0,0) \in S \neq \phi$

equivalently

A linear system is homogeneous if $\vec{0}$ is a solution