Vectors and Vector Spaces

For the beginning of the course, we will define a *vector* and *vector space* in this way (this is not the most abstract or the best definition, but it is what we will start with):

Definition 1. A *real vector* is a column of n real numbers. We call such a vector an n *dimensional real vector*. The set of all n dimensional vectors is $\mathbb{R}^n$.

In this series of videos, we will concentrate mostly on the $n = 2$ and $n = 3$ cases. The reason is that most Math 308 classes only use 2 and 3 dimensional vectors and because once the basic application of linear algebra to differential equations is understood, you can come back to the subject after you have had a proper linear algebra class.

Example. Here are some examples:

(1) The vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a vector in $\mathbb{R}^2$.

(2) The vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is a vector in $\mathbb{R}^3$.

(3) $\begin{pmatrix} 1 \\ 2+3i \\ 0 \end{pmatrix}$ is not in $\mathbb{R}^3$ since the second entry, $2 + 3i$, is not a real number.

Definition 2. A *complex vector* is a column of n complex numbers. We call such a vector an n *dimensional complex vector*. The set of all n dimensional complex vectors is $\mathbb{C}^n$.

Example. All the examples in the previous example are complex vectors.

Vectors and Vector Spaces

In the study of differential equations, the vectors in question are functions of some variable – for example – t. This means that each entry in the vector is a function. For example, the following vectors are typical of those that come up in differential equations:

(1) $\begin{pmatrix} t \\ t^2 \end{pmatrix}$

(2) $\begin{pmatrix} e^t \\ e^{2t} \\ 1 \end{pmatrix}$

(3) $e^t \begin{pmatrix} \cos t + \sin t \\ \cos t - \sin t \\ \cos t \end{pmatrix}$

These are all real vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ for fixed values of t. But it is a (potentially) different vector for each value of t. Some people call vectors like these "vector–valued functions."

Vectors and Vector Spaces

There are two important ways to create new vectors out of old vectors that we now discuss. The first that we discuss is *scalar multiplication.* Given a vector $\boldsymbol{v} \in \mathbb{R}^n$, and a real or complex number c (real and complex numbers are called *scalars*) the product $c\boldsymbol{v}$ is defined as follows:

$$c\boldsymbol{v} = c\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix}.$$

That is, each entry of $\boldsymbol{v}$ is just multiplied by the scalar c. As you will learn when you take an actual linear algebra class (and as you might know from Physics–related applications), multiplying a vector by a scalar has the geometric interpretation of "stretching" the vector. While geometric interpretations are always helpful for the intuition they bring, this won't be a focus of this mini course.

Example. Compute $3\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

Solution. We just multiply each entry by 3:

$$3\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix}.$$

Example. Compute $e^t\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$.

Solution. We just multiply each entry by e^t:

$$e^t\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}.$$

Vectors and Vector Spaces

The next way to combine vectors that we discuss is adding two vectors together. Let $\boldsymbol{v}, \boldsymbol{w}$ be two vectors in $\mathbb{R}^n$. Then we can define the sum of these vectors as:

$$\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}.$$

That is, we just add the vectors "entry–wise". Note that two add two vectors together, they must both be in $\mathbb{R}^n$. That is, addition of a vector in $\mathbb{R}^2$ and a vector in $\mathbb{R}^3$ is not defined.

Example. Find the sum $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

Solution.

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 0+0 \\ -1+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.$$

Example. Find the sum $e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ and $e^{2t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$.

Solution.

$$e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + e^{2t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} = \begin{pmatrix} e^t \cos t \\ e^t \sin t \end{pmatrix} + \begin{pmatrix} e^{2t} \cos 2t \\ e^{2t} \sin 2t \end{pmatrix} = \begin{pmatrix} e^t \cos t + e^{2t} \cos 2t \\ e^t \sin t + e^{2t} \sin 2t \end{pmatrix}.$$

Vectors and Vector Spaces

The last example is an example of something called a *linear combination* – this is when we combine both scalar multiplication and vector addition into one operation. In general, if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are vectors in $\mathbb{R}^n$ and $c_1, \ldots, c_m$ are scalars, then we can form the linear combination:

$$c_1\boldsymbol{v}_1 + \cdots + c_m\boldsymbol{v}_m = c_1 \begin{pmatrix} a_1^1 \\ \vdots \\ a_n^1 \end{pmatrix} + \cdots + c_m \begin{pmatrix} a_1^m \\ \vdots \\ a_n^m \end{pmatrix} = \begin{pmatrix} c_1 a_1^1 \\ \vdots \\ c_1 a_n^1 \end{pmatrix} + \cdots + \begin{pmatrix} c_m a_1^m \\ \vdots \\ c_m a_n^m \end{pmatrix} = \begin{pmatrix} c_1 a_1^1 + \vdots + c_m a_1^m \\ \vdots \\ c_1 a_n^1 + \cdots + c_m a_n^m \end{pmatrix}.$$

Example. Compute $2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 10 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Solution.

$$2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 10 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} -30 \\ -6 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -26 \\ -1 \end{pmatrix}.$$

Motivation from Systems of Equations

Oftentimes (and we will see specific applications to ODEs later in the course), we want to solve systems of equations like:

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}.$$

This means that we want to find all c_1, c_2, c_3 that satisfy this equation. So there are two questions to ask: (1) Is there a solution? and (2) If there is, is the solution unique?
For this particular problem:

$$\begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

And so one solution is:

$$c_1 = 3 \qquad c_2 = -1 \qquad c_3 = 2.$$

Next, we need to answer: is there more than one solution? If c_1, c_2, c_3 satisfy the equation then the top equation implies $c_2 = 2 - c_1$ and the middle equation implies $c_3 = 5 - c_1$. We can now plug this in the bottom equation to get:

$$\begin{aligned} 2 &= c_1 - c_2 - c_3 \\ &= c_1 - (2 - c_1) - (5 - c_1) \\ &= 3c_1 - 7 \end{aligned}$$

so $c_1 = 3$ and so $c_2 = 2 - c_1 = -1$ and $c_3 = 5 - c_1 = 2$. So these are the only solutions.
The questions above are called "existence" and "uniqueness" questions and are related to the concepts of "spanning sets" and "linear independence" that we will discuss in this lecture.

Linear Independence

Definition 3. A set of vectors, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ in $\mathbb{R}^n$ is said to be *linearly independent* if

the only solution to:

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_m\boldsymbol{v}_m = \mathbf{0}$$

is $c_1 = c_2 = \cdots = c_m = 0$.
Observe that if $c_1 = \cdots = c_m = 0$, then this is a solution to the above equation. So a set of vectors is linear independent if this is the only solution. The terminology can be explained as follows. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are linearly dependent, then there is some choice of scalars $c_1, \ldots c_m$ not all zero such that $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_m\boldsymbol{v}_m = \mathbf{0}$. If we assume that $c_1 \neq 0$ then:

$$\boldsymbol{v}_1 = -\frac{1}{c_1}(c_2\boldsymbol{v}_2 + \cdots + c_m\boldsymbol{v}_m).$$

That is, $\boldsymbol{v}_1$ *depends* on the other vectors. Another way to define linear independence is:

Definition 4. A set of vectors is *linearly dependent* if one vector can be written as a linear combination of the others. And a set of vectors is *linearly independent* if they are not linearly dependent.

Let's look at a few examples.

Example. Determine if the vectors $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}$ are linearly independent.

We need to determine the solutions to the equation:

$$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

This is the same as:

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 0. \end{aligned}$$

The second equation says $c_1 = c_2$. Plugging this into the first equation gives:

$$2c_1 = 0.$$

This means $c_1 = 0$ and since $c_1 = c_2$, this means $c_2 = 0$. So the only solution is $c_1 = c_2 = 0$. Thus the vectors in question are linearly independent.

Linear Independence

Example. Determine if the vectors $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-2\\0\end{pmatrix}$ are linearly independent.

Similar to the above, we want to solve the equation:

$$c_1 \begin{pmatrix}1\\1\end{pmatrix} + c_2 \begin{pmatrix}1\\-1\end{pmatrix} + c_3 \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

This is the same as:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 - c_2 + 0 &= 0. \end{aligned}$$

The second equation says that $c_1 = c_2$. Inserting this into the first equation gives:

$$2c_1 + c_3 = 0.$$

This says that $c_1 = -\frac{1}{2}c_3$ but doesn't give any other restriction. Thus, any set of coefficients c_1, c_2, c_3 that satisfies $c_2 = c_1 = -\frac{1}{2}c_3$ is a solution. For example, if we take $c_3 = -2$ then $c_1 = c_2 = 1$ is a solution. Indeed:

$$\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix} - 2\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\0\end{pmatrix}.$$

So these vectors are not linearly independent.

Linear Independence

Example. Show that the vectors

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

are linearly independent.

We need to solve:

$$c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\\-1\end{pmatrix} + c_3\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

This is the same as:

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + c_3 &= 0 \\ c_1 - c_2 - c_3 &= 0. \end{aligned}$$

The first two equations say that $c_2 = c_3 = -c_1$. Inserting this into the third equation gives $c_1 + c_1 + c_1 = 0$ whence c_1 and thus c_2 and c_3 are all zero. So, the only solution is $c_1 = c_2 = c_3 = 0$ and so these vectors are linearly independent.

Linear Independence

Returning to the example:

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

observe this can be written as:

$$(c_1 - 3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (c_2 - (-1)) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + (c_3 - 2) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since we know these vectors are linearly independent, it follows that:

$$c_1 - 3 = 0 \qquad c_2 - (-1) = 0 \qquad c_3 - 2 = 0.$$

Spanning Sets and Basis

The concepts of "spanning set" and "basis" are related to the concept of "linear independence". A set of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ in $\mathbb{R}^n$ is a spanning set for $\mathbb{R}^n$ if every vector in $\mathbb{R}^n$ can be written as a linear combination of these vectors.

Example. Determine if $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are a spanning set for $\mathbb{R}^2$.

Can we solve:

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

for all values of x, y? (Here, x and y are "known" but arbitrary and the c_1, c_2 are "unknown". The "bottom" equation implies that $c_1 = y$. Inserting this into the "top" equation gives $y + c_2 = x$ and so $c_2 = x - y$. So,

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (x - y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since this is true for all x, y it means these vectors form a spanning set.

Spanning Sets and Basis

A general fact is that n linearly independent vectors in $\mathbb{R}^n$ form a spanning set. In addition, a set of vectors that are (1) linearly independent and (2) a spanning set is called a *basis*.

Example. Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ form a basis for $\mathbb{R}^3$.

We have seen that they are linearly independent, there are three of them, and they are in $\mathbb{R}^3$ so they form a spanning set.