Vectors and Vector Spaces

For the beginning of the course, we will define a *vector* and *vector space* in this way (this is not the most abstract or the best definition, but it is what we will start with):

Definition 1.

In this series of videos, we will concentrate mostly on the $n = 2$ and $n = 3$ cases. The reason is that most Math 308 classes only use 2 and 3 dimensional vectors and because once the basic application of linear algebra to differential equations is understood, you can come back to the subject after you have had a proper linear algebra class.

Example. Here are some examples:

(1)

(2)

(3)

Definition 2.

Example. All the examples in the previous example are complex vectors.

Vectors and Vector Spaces

In the study of differential equations, the vectors in question are functions of some variable – for example – t. This means that each entry in the vector is a function. For example, the following vectors are typical of those that come up in differential equations:

(1)

(2)

(3)

Vectors and Vector Spaces

There are two important ways to create new vectors out of old vectors that we now discuss. The first that we discuss is *scalar multiplication*. Given a vector $\boldsymbol{v} \in \mathbb{R}^n$, and a real or complex number c (real and complex numbers are called *scalars*) the product $c\boldsymbol{v}$ is defined as follows:

$$c\boldsymbol{v} = c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix}.$$

That is, each entry of $\boldsymbol{v}$ is just multiplied by the scalar c. As you will learn when you take an actual linear algebra class (and as you might know from Physics–related applications), multiplying a vector by a scalar has the geometric interpretation of "stretching" the vector. While geometric interpretations are always helpful for the intuition they bring, this won't be a focus of this mini course.

Example. Compute $3 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

Solution.

Example. Compute $e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$.

Solution.

Vectors and Vector Spaces

The next way to combine vectors that we discuss is adding two vectors together. Let $\boldsymbol{v}, \boldsymbol{w}$ be two vectors in $\mathbb{R}^n$. Then we can define the sum of these vectors as:

$$\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}.$$

That is, we just add the vectors "entry–wise". Note that two add two vectors together, they must both be in $\mathbb{R}^n$. That is, addition of a vector in $\mathbb{R}^2$ and a vector in $\mathbb{R}^3$ is not defined.

Example. Find the sum $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

Solution.

Example. Find the sum $e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ and $e^{2t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$.

Solution.

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The last example is an example of something called a *linear combination* – this is when we combine both scalar multiplication and vector addition into one operation. In general, if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are vectors in $\mathbb{R}^n$ and $c_1, \ldots, c_m$ are scalars, then we can form the linear combination:

$$c_1\boldsymbol{v}_1 + \cdots + c_m\boldsymbol{v}_m = c_1\begin{pmatrix} a_1^1 \\ \vdots \\ a_n^1 \end{pmatrix} + \cdots + c_m\begin{pmatrix} a_1^m \\ \vdots \\ a_n^m \end{pmatrix} = \begin{pmatrix} c_1a_1^1 \\ \vdots \\ c_1a_n^1 \end{pmatrix} + \cdots + \begin{pmatrix} c_ma_1^m \\ \vdots \\ c_ma_n^m \end{pmatrix} = \begin{pmatrix} c_1a_1^1 + \vdots + c_ma_1^m \\ \vdots \\ c_1a_n^1 + \cdots + c_ma_n^m \end{pmatrix}.$$

Example. Compute $2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 10 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Solution.

Motivation from Systems of Equations

Oftentimes (and we will see specific applications to ODEs later in the course), we want to solve systems of equations like:

For this particular problem:

$$\begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} =$$

And so one solution is:

$$c_1 = \qquad c_2 = \qquad c_3 = .$$

Next, we need to answer: is there more than one solution?

The questions above are called "existence" and "uniqueness" questions and are related to the concepts of "spanning sets" and "linear independence" that we will discuss in this lecture.

Linear Independence

Definition 3. A set of vectors, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ in $\mathbb{R}^n$ is said to be *linearly independent* if

Let's look at a few examples.

Example. Determine if the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent.

Linear Independence

Example. Determine if the vectors $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-2\\0\end{pmatrix}$ are linearly independent.

Linear Independence

Example. Show that the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

are linearly independent.

Linear Independence

Returning to the example:

$$c_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + c_2 \begin{pmatrix}1\\0\\-1\end{pmatrix} + c_3 \begin{pmatrix}0\\1\\-1\end{pmatrix} = 3 \begin{pmatrix}1\\1\\1\end{pmatrix} - 1 \begin{pmatrix}1\\0\\-1\end{pmatrix} + 2 \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

observe this can be written as:

$$(c_1 - 3) \begin{pmatrix}1\\1\\1\end{pmatrix} + (c_2 - (-1)) \begin{pmatrix}1\\0\\-1\end{pmatrix} + (c_3 - 2) \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Since we know these vectors are linearly independent, it follows that:

$$c_1 - 3 = 0 \qquad c_2 - (-1) = 0 \qquad c_3 - 2 = 0.$$

Spanning Sets and Basis

The concepts of "spanning set" and "basis" are related to the concept of "linear independence". A set of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ in $\mathbb{R}^n$ is a spanning set for $\mathbb{R}^n$ if every vector in $\mathbb{R}^n$ can be written as a linear combination of these vectors.

Example. Determine if $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are a spanning set for $\mathbb{R}^2$.

Spanning Sets and Basis

A general fact is that n linearly independent vectors in $\mathbb{R}^n$ form a spanning set. In addition, a set of vectors that are (1) linearly independent and (2) a spanning set is called a *basis*.

Example. Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ form a basis for $\mathbb{R}^3$.